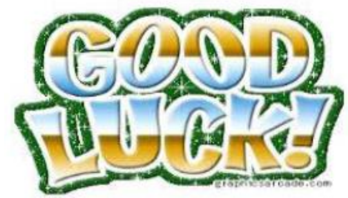
Reexam Mathematical Physics, Prof. G. Palasantzas

- Total number of points 100
- 10 points for coming to the exam
- Justify briefly your answers for all problems



<u>Problem 1 (10 points)</u> Show that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ with $x \in (-\infty, +\infty)$

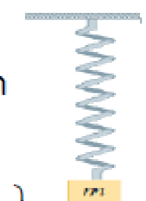
<u>Problem 2 (15 points)</u>

Determine weather the series $\sum_{n=1}^{\infty} \frac{\cos 3n}{1 + (1.2)^n}$ is convergent or divergent.

<u>Problem 3 (20 points)</u>

Suppose a mass m is attached to a spring with spring constant k, and let $k = m\omega^2$. If an external force $F(t) = F_o \cos(\omega_o t)\ (\omega_o \neq \omega)$ is applied, then we have the equation of motion for non-zero dissipation (c>0):

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \quad (1)$$



Show that a particular solution of Eq. (1) $(c^2 - 4mk < 0)$ is given by: $x_p(t) = \left\{ \frac{F_o}{\sqrt{(k - m\omega_0^2)^2 + (c\omega_0)^2}} \right\} \cos(\omega_0 t + \delta)$

Tip: $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(\omega t + \delta)$ with $A = \sqrt{c_2^{\,2} + c_1^{\,2}}$ and $\tan\delta = -c_2 / c_1$ $\quad \tan\delta = -c\omega_0/(k - m\omega_0^2)$

Problem 4 (15 points)

Find the sine series solution of the second order differential equation $\frac{d^2X}{dt^2} + 10X = f(t)$ with *f(t)* an odd function [*f(-t)=-f(t)*] and boundary conditions X(t=0)=X(t=L)=0.

Tip : The Fourier sine expansion has the general form $Y(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x / L)$, $b_n = (2/L)\int_0^L Y(x)\sin(n\pi x/L)dx$, for $x \in [0, L]$

Problem 5 (10 points)

Consider the Fourier transform definitions $F(k) = \int_{-\infty}^{+\infty} F(x)e^{-i2\pi kx}dx$ and $\delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx}dx$

Calculate the Fourier transform of $F(x) = sin^2(\varepsilon x) + \cos(\varepsilon x)$

Problem 6 (20 points)

(a: 10 points) Calculate the analytic form of the function $G(\omega)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} t^2 e^{-|\omega t|}e^{-2\pi ikt}\cos^2(8\pi k)dtdk$ with ω a real number.

(b: 10 points) Consider the boundary value problem for the one-dimensional heat equation for a bar of length L with zero-temperature ends. The general solution u(x, t) (with t≥0 and 0≤x≤L) has the form:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\frac{n\pi}{L}x \ , \ \lambda_n = \frac{cn\pi}{L}$$

Calculate the solution u(x, t) for t>0 if u(x, t=0)=f(x)=2sin(6πx/L)cos(3πx/L).

Problem 1

We form the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, and we show that it is covergent using the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1 \quad \text{with } x \in (-\infty, +\infty)$$

Therefore, since the series is convergent we have: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^n}{n!} = 0$

Problem 2

Check convergence of→ $\sum_{n=1}^{\infty} \frac{\cos 3n}{1+(1.2)^n}$

$|a_n| = \left|\frac{\cos 3n}{1+(1.2)^n}\right| \leq \frac{1}{1+(1.2)^n} < \frac{1}{(1.2)^n} = \left(\frac{5}{6}\right)^n$, so $\sum_{n=1}^{\infty} |a_n|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$ $\left[r = \frac{5}{6} < 1\right]$. It follows that $\sum_{n=1}^{\infty} a_n$ converges

Problem 3

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0\cos\omega_0 t \quad (1)$$

$$X_p(t) = A(\omega_0)\cos(\omega_0 t) + B(\omega_0)\sin(\omega_0 t)$$

substitute in (1) $\Rightarrow$

$$m(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + (-AC\omega_0\sin\omega_0 t + BC\omega_0\cos\omega_0 t) + k(A\cos\omega_0 t + B\sin\omega_0 t) = F_0\cos\omega_0 t \Rightarrow$$

$$[(k-m\omega_0^2)A + CB\omega_0]\cos\omega_0 t + [(k-m\omega_0^2)B - CA\omega_0]\sin\omega_0 t = F_0\cos\omega_0 t \quad (2)$$

(2) $\Rightarrow$

$$(k-m\omega_0^2)A + CB\omega_0 = F_0 \quad (3)$$

$$(k-m\omega_0^2)B - CA\omega_0 = 0 \quad (4)$$

(4) $\Rightarrow$

$$B = A\frac{C\omega_0}{k-m\omega_0^2} \quad (5)$$

substitute (5) into (3) $\Rightarrow$

$$A = \frac{F_0(k-m\omega_0^2)}{(k-m\omega_0^2)^2 + (C\omega_0)^2} \quad (6)$$

$$\rightarrow B = \frac{F_0 C\omega_0}{(k-m\omega_0^2)^2 + (C\omega_0)^2} \quad (7)$$

The particular solution can be written as $X_p(t) = \tilde{A}(\omega_0)\cos(\omega_0 t + \delta)$

with $\tilde{A}(\omega) = \sqrt{A^2+B^2} = \dfrac{F_0}{\sqrt{(k-m\omega_0^2)^2 + (C\omega_0)^2}}$

$$\tan\delta = -\frac{B}{A} = -\frac{C\omega_0}{k-m\omega_0^2} \Rightarrow$$

$$\delta = \tan^{-1}\left\{\frac{C\omega_0}{m\omega_0^2 - k}\right\}$$

Problem 4

Find the solution* of the differential equation

$y'' + my = f(x)$, m is integer, $f(x) = -f(-x)$ [odd function]

Under the conditions $y(0) = y(x=L) = 0$

* Sine solution.

$$y(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (1)$$

Take $f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$, $C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$y''(x) = \sum_{n=1}^{\infty} \left(-\left(\frac{n\pi}{L}\right)^2\right) b_n \sin\left(\frac{n\pi x}{L}\right) \quad (2)$$

substitution into $y'' + my = f(x)$ gives

$$\sum_{n=1}^{\infty}\left[-b_n\left(\frac{n\pi}{L}\right)^2\right]\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} m b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\sum_{n=1}^{\infty}\left\{b_n\left[m - \left(\frac{n\pi}{L}\right)^2\right] - C_n\right\}\sin\left(\frac{n\pi x}{L}\right) = 0 \quad \forall x \in [0, L]$$

$$b_n\left[m - \left(\frac{n\pi}{L}\right)^2\right] - C_n = 0 \Rightarrow$$

$$b_n = \frac{C_n}{m - \left(\frac{n\pi}{L}\right)^2} \qquad \left(m \neq \left(\frac{n\pi}{L}\right)^2\right)$$

$$\boxed{y(x) = \sum_{n=1}^{\infty} \frac{C_n}{m - \left(\frac{n\pi}{L}\right)^2}\sin\left(\frac{n\pi x}{L}\right)}$$

Set m=10
replace the variable x with t.

Problem 5

5

$$F(x) = -\frac{1}{4}\left(e^{i\varepsilon x} - e^{-i\varepsilon x}\right)^2 + \frac{1}{2}\left(e^{i\varepsilon x} + e^{-i\varepsilon x}\right)$$

$$\Rightarrow F(x) = \frac{1}{2}e^{i\varepsilon x} + \frac{1}{2}e^{-i\varepsilon x} - \frac{1}{4}e^{i2\varepsilon x} - \frac{1}{4}e^{-i2\varepsilon x} + \frac{1}{2}$$

$$F(k) = \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} e^{-i\varepsilon\pi x\left(k - \frac{\varepsilon}{2\pi}\right)}}_{\delta\left(k-\frac{\varepsilon}{2\pi}\right)} dx + \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi x\left(k + \frac{\varepsilon}{2\pi}\right)}}_{\delta(k+\varepsilon/2\pi)} dx$$

$$- \frac{1}{4}\underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi x\left(k - \frac{\varepsilon}{\pi}\right)}}_{\delta\left(k-\frac{\varepsilon}{\pi}\right)} dx - \frac{1}{4}\underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi x\left(k + \frac{\varepsilon}{\pi}\right)}}_{\delta\left(k+\frac{\varepsilon}{\pi}\right)} dx + \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi kx}}_{\delta(k)} dx$$

$$\Rightarrow F(k) = \frac{1}{2}\left[\delta\left(k-\frac{\varepsilon}{2\pi}\right) + \delta\left(k+\frac{\varepsilon}{2\pi}\right)\right] - \frac{1}{4}\left[\delta\left(k-\frac{\varepsilon}{\pi}\right) + \delta\left(k+\frac{\varepsilon}{\pi}\right)\right] + \frac{1}{2}\delta(k)$$

Problem 6

(6a) $\cos^2(8\pi k) = \frac{1}{4}\left[e^{i16\pi k} + e^{-i16\pi k} + 2\right]$

$$\Rightarrow G(\omega) = \int_{-\infty}^{+\infty} dt\, t^2 e^{-|\omega t|} \Big\{ \frac{1}{4} \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi k(t-8)}\, dk}_{\delta(t-8)} + \frac{1}{4} \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi k(t+8)}\, dk}_{\delta(t+8)} + \frac{1}{2} \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi kt}}_{\delta(t)} \Big\}$$

$$\Rightarrow G(\omega) = \int_{-\infty}^{+\infty} t^2 e^{-|\omega t|} \Big\{ \frac{1}{4}\delta(t-8) + \frac{1}{4}\delta(t+8) + \frac{1}{2}\delta(t) \Big\}\, dt$$

$$\Rightarrow \underline{G(\omega) = 32\, e^{-|8\omega|}}$$

(6b) If we set $t=0$ to the general solution

we get $u(x,t=0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$ (1)

rewrite the give $u(x,t=0) = f(x)$ as a sum of sines.

$u(x,t=0) = \sin\left(\frac{9\pi}{L}x\right) + \sin\left(\frac{3\pi}{L}x\right)$ (2)

If we compare (1) & (2) then only the terms $n=9$ and $n=3$ are non-zero in (1)

Thus we have $B_9 = B_3 = 1$, $B_n = 0$ $(n \neq 3, 9)$

$$\Rightarrow \underline{u(x,t>0) = e^{-\left(\frac{c3\pi}{L}\right)^2 t} \sin\left(\frac{3\pi}{L}x\right) + e^{-\left(\frac{c9\pi}{L}\right)^2 t} \sin\left(\frac{9\pi}{L}x\right)}$$